System reliability and component importance when components can be swapped upon failure

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Abstract

Resilience of systems to failures during functioning is of great practical importance. One of the strategies that might be considered to enhance reliability and resilience of a system is swapping components when a component fails, so replacing it by another component from the system which is still functioning. This paper studies this scenario, particularly with the use of the survival signature to quantify system reliability, where it is assumed that such a swap of components requires these components to be of the same type. We examine the effect of swapping components on a reliability importance measure for the specific components and we also consider the joint reliability importance of two components. Such swapping of components may be an attractive means towards more resilient systems, and could be an alternative to adding more components to achieve redundancy of repair and replacement activities.

Keywords: Importance measures, joint reliability importance, resilience, survival signature, swapping components, system reliability

1. Introduction

With the need for highly reliable systems, there are many possibilities to make a basic system more reliable, or more resilient to possible faults. It may be possible to add component redundancy or make individual components more reliable. In addition, one may be able to repair failed components or replace them with new ones. In this paper, we consider a quite straightforward way in which some systems may become resilient to component failure, namely the possibility to replace a failed component by another component in the system which has not yet failed, so in effect swapping components. This is logically restricted to components which are of the same type, hence it is likely that only some swapping opportunities exist in a system. It seems that increase in system reliability through such component swapping has not received much attention in the literature, yet in some scenarios it can be an attractive opportunity to prevent a system from failing. In practice, this could enable preparation of substantial repair activities, or it may be deemed to leave the system reliable enough to complete its mission. Scenarios where swapping of components may be an option can include the following examples. Aerospace systems with multiple computers on board, where some computers tasked with minor functions can be prepared to take over crucial functions in case another computer fails, or lighting systems where multiple locations must be provided with light under contract, but where partial lighting at any location may be sufficient to meet the contractual requirements. One can also think about a transport system where parts of one mode of transport can be used to keep another one running, or an organisation where employees can be trained to take over some functioning of others in case of unexpected absence.

It should be emphasized that swapping a component, upon failure, with another component from the system, differs from the well-studied scenarios of using cold or warm standby components, or adding components in parallel for to achieve increased reliability. When we replace a failed component with a functioning component that was already in the system, the subsystem in which the latter component was originally placed becomes less reliable. For example, as we will see in Example 1 in Section 2, we might replace a critical component upon failure by another component that is part of a subsystem consisting of three components in parallel; after such a swap that subsystem is reduced to a subsystem consisting only of two components in parallel. One can also compare the kind of component swapping studied in this paper to a minimal repair [1], in that the failure time distribution of the component does not change, but this is combined with a change in the overall structure of the system due to the functioning component being removed elsewhere.

In this paper we consider the effect of defined component swap possibilities on the total system reliability function. We also consider the importance of individual components, which can be strongly affected by opportunities to swap them, and the joint reliability importance of two components.

Quantification of system reliability has traditionally been based on the structure function [2, 3, 4]. Samaniego [5] introduced the system signature as a tool for reliability assessment for systems consisting of components of a single type, which means that their failure time distributions are exchangeable [6, 7]. Samaniego's signature can be regarded as a summary of the structure function that is sufficient to derive the system reliability function if the failure times of all the system's components are exchangeable. The main attraction of this signature is that it enables separation of aspects of the system structure and the components failure times distribution, which simplifies a range of reliability related problems such as stochastic comparison of different system structures and inference on the system reliability from component failure data.

The major drawback of Samaniego's signature is that it can only be applied to systems with a single type of components, which is quite rare for real-world systems and prevents the method to be used for analysis of networks [8]. To overcome this limitation, Coolen and Coolen-Maturi [9] introduced the survival signature as an alternative tool for system reliability quantification. This is also a summary of the system structure function which is sufficient for a range of reliability computations and inferences, including derivation of the system reliability function, and crucially it can be used for systems with multiple types of components. The only requirement is that failure times of components of the same type are exchangeable. Components of different types can be dependent, of course any such dependence must be modelled, for example through the use of copulas [10] or the use of multivariate failure time models including dependence [11, 12]. In this paper, to present the swapping opportunities without further complications, we throughout assume that components of different types have independent failure times, and in addition we assume that components of the same type are conditionally independent and identically distributed. These assumptions can be relaxed without difficulty, such relaxation can of course alter the effect of enabled swaps on the overall system reliability.

The survival signature is closely linked to Samaniego's system signature for systems with a single type of components, and it is particularly useful for larger systems with only a few different types of components. Recently, the survival signature has attracted considerable interest from researchers in reliability, who have considered both mathematical properties and aspects of application, including statistical inference [13, 8], comparison of different systems [14] and fast simulation methods [15]. In this paper, we will consider scenarios where some components of the same type can be swapped, and we use the survival signature to derive the corresponding system reliability.

In addition to considering the effect of component swapping on the system reliability function, we also consider how such possible swaps affect component importance as reflected through importance measures. Such measures are frequently used as tools to evaluate and rank the impact of specific components on the system reliability [16]. Quite many different important measures have been introduced, such as Birnbaums measure, the improvement potential measure, the risk achievement worth, the risk reduction worth, the criticality importance measure and Fussell-Veselys measure of importance [17]. The objective of most of these measures is to assess and quantify which components are more critical to system failure or more important to system reliability improvement. The importance of a component is, of course, largely defined through its function in a system, hence we can expect that the ability to swap components can have a strong effect on the importance of a component. We also consider the joint importance of two components for the reliability of a system. Hong and Lie [18] defined the joint reliability importance (JRI) as a measure of how two components in a system interact in contributing to system reliability. This definition was extended in several ways. For example, Armstrong [19] presents a joint importance measure for dealing with the statistical dependence between components and Wu [20] generalized JRI to multi-state systems. Recently, Eryilmaz et al [12] have presented general results on marginal and joint reliability importance for components with dependent failure time distributions, they also used the concept of survival function.

In section 2 of this paper, we first provide a brief introductory overview of the concept of the survival signature, followed by new results for system reliability if components can be swapped upon failure of a component. In section 3 we consider the impact of possible component swapping on a reliability importance measure for an individual component, followed in section 4 by attention to joint reliability importance of two components. In each section, we illustrate our approach via examples. We end the paper with some concluding remarks.

2. Swapping components

Consider a system that consists of m components of $K \geq 2$ different types, with m_k components of type $k \in \{1, 2, ..., K\}$, so $\sum_{k=1}^{K} m_k = m$. Assume that the random failure times of components of the same type are exchangeable, while full independence is assumed for the random failure times of components of different types. As mentioned in Section 1, this independence assumption simplifies the presentation in this paper, but the survival signature can also be applied without it, hence the effect of swapping of components can be studied in a similar way without this assumption. Let the state vector $\underline{x}^k = \left(x_1^k, x_2^k, ..., x_{m_k}^k\right) \in \{0,1\}^{m_k}$ represent the state of the system components of type k, with $x_i^k = 1$ if the *ith* component of type k functions and $x_i^k = 0$ if not. The labelling of the components is arbitrary but must be fixed to define $\underline{x}^k$. Let $\underline{x} = \left(\underline{x}^1, \underline{x}^2, ..., \underline{x}^K\right) \in \{0,1\}^m$ be the state vector for the overall system. The structure function $\phi : \{0,1\}^m \to \{0,1\}$, defined for all possible $\underline{x}$, takes the value 1 for a particular state vector $\underline{x}$ if the system functions and 0 if the system does not function for the state vector $\underline{x}$. The survival signature is denoted by $\Phi\left(l_1, l_2, ...l_K\right)$ and represents the probability that the system functions, given that exactly l_k of type k components function, for $l_k = 0, 1, ..., m_k$, for each $k = 1, 2, ..., K$ [9].

There are $\binom{m_k}{l_k}$ state vectors $\underline{x}^k$ with exactly l_k of the m_k components $x_i^k = 1$, so with $\sum_{i=1}^{m_k} x_i^k = l_k$. We denote the set of these state vectors for components of type k by S_l^k. Let $S_{l_1,...,l_K}$ denote the set of all state vectors for the whole system for which $\sum_{i=1}^{m_k} x_i^k = l_k$, for $k = 1, 2, ..., K$. Because of the assumption that the failure times of m_k components of type k are exchangeable, all the state vectors $\underline{x}^k \in S_l^k$ are equally likely to occur. Thus $\Phi\left(l_1, l_2, ...l_K\right)$ can be calculated by

$$\Phi\left(l_1, l_2, ...l_K\right) = \left(\prod_{k=1}^{K} \binom{m_k}{l_k}^{-1}\right) \times \sum_{\underline{x} \in S_{l_1,...,l_K}} \phi\left(\underline{x}\right)$$

Let $C_t^k \in \{0, 1, ..., m_k\}$ denote the number of type k components in the system that function at time $t > 0$. Using the assumed independence of failure times of components of different types, the probability that the system functions at time $t > 0$ is

$$P(T_S > t) = \sum_{l_1=0}^{m_1} ... \sum_{l_K=0}^{m_K} \left[\Phi(l_1, ...l_K) \prod_{k=1}^{K} P(C_t^k = l_k)\right]$$

Note that, if one would not assume independence of the failure times of components of different types, then the product of the marginal probabilities for individual events $C_t^k = l_k$ in this formula would be replaced by the joint probability of these events, from which point a model must be assumed for this joint probability as mentioned in Section 1. Henceforth we assume, in addition to exchangeability of failure times of components of the same type, that these failure times are conditionally independent and identically distributed, with the probability distribution for the failure time of components of type k specified by the cumulative distribution function (CDF) $F_k(t)$. This leads to

$$P(T_S > t) = \sum_{l_1=0}^{m_1} \cdots \sum_{l_K=0}^{m_K} \left[\Phi(l_1, \ldots l_K) \prod_{k=1}^{K} \left(\binom{m_k}{l_k} [F_k(t)]^{m_k - l_k} [1 - F_k(t)]^{l_k} \right) \right]$$

The survival signature takes into account the structure of the system, and as is clear from the above equations this information is separated from the failure time distributions of the system components. We now consider the situation where some components can be swapped upon failure of a component. Actually, throughout this paper we assume that there are fixed swapping rules, which prescribe upon failure of a component precisely which other component takes over its role in the system, if possible and if the other component is still functioning. We further assume here that such a swap of components takes neglectable time and does not affect the functioning of the component that changes its role in the system nor its remaining time till failure. Under these assumptions, the effect of such a component swap can be reflected through the system structure function, and hence it can be taken into account for computation of the system reliability through the survival signature. We illustrate this in examples below, after we have introduced the required notation, and we discuss it further at the end of this section.

For a regime of specified swaps that will occur if specific components fail, let $\phi_C(\underline{x})$ denote the system structure function. Compared to the system's structure function without swapping opportunities, $\phi(\underline{x})$, ϕ_C will typically be equal to 1 for some $\underline{x}$ for which ϕ was equal to 0, reflecting the benefit from component swapping upon failure of a component. Let $\Phi_C(l_1, l_2, \ldots l_K)$ denote the survival signature given the defined swapping regime is in place, so

$$\Phi_C(l_1, l_2, \ldots l_K) = \left(\prod_{a=1}^{K} \binom{m_k}{l_k}^{-1} \right) \times \sum_{\underline{x} \in S_{l_1, \ldots, l_K}} \phi_C(\underline{x}).$$

Let T_S^* denote the random system failure time with the specified swapping regime in place. Then

$$P(T_S^* > t) = \sum_{l_1=0}^{m_1} ... \sum_{l_K=0}^{m_K} \left[\Phi_C(l_1, ...l_K) \prod_{k=1}^{K} \left(\binom{m_k}{l_k} [F_k(t)]^{m_k - l_k} [1 - F_k(t)]^{l_k} \right) \right]$$

It is important to notice here that the swapping regime is entirely reflected in the system survival signature. Crucially, the components have remained the same failure time distributions and the same assumptions apply, that is failure times of components of the same type remain conditionally independent and identically distributed, and failure times of components of different types remain independent. The increase in reliability caused by the swapping regime, when compared to the system without possible swapping, is given by

$$P(T_S^* > t) - P(T_S > t) =$$
$$\sum_{l_1=0}^{m_1} ... \sum_{l_K=0}^{m_K} \left[\{\Phi_C(l_1, ...l_K) - \Phi(l_1, ...l_K)\} \prod_{k=1}^{K} \left(\binom{m_k}{l_k} [F_k(t)]^{m_k - l_k} [1 - F_k(t)]^{l_k} \right) \right]$$

Hence, as long as a swapping regime leads to increase of the survival signature, for at least one of its values, it will be of benefit for the overall system reliability. It is also obvious that a series system can never benefit from such swapping, simply because it only functions if all of its components function. This is reflected by the fact that for a series system the two survival signatures considered here are always equal. The above result for the difference of the reliability of the system with and without possible swapping, ensures that some relevant computations, for example for importance measures as presented in Sections 3 and 4, are quite straightforward.

To illustrate the above way to reflect the effect of a component swapping regime, we first present a simple example with only one possible component swap. This will be followed by a more extensive example in which we compare different possible of components swaps.

2.1. Example 1

Consider the system in Figure 1 which consists of four components of two types, $m_1 = m_2 = 2$. We want to examine the reliability of this system in the case that, if component A fails but component B still functions, component B will take over the role of component A. Of course, this swap only has a positive effect on the system reliability if, at the time of the swap, at least one of components C and D also still functions. So, the system's

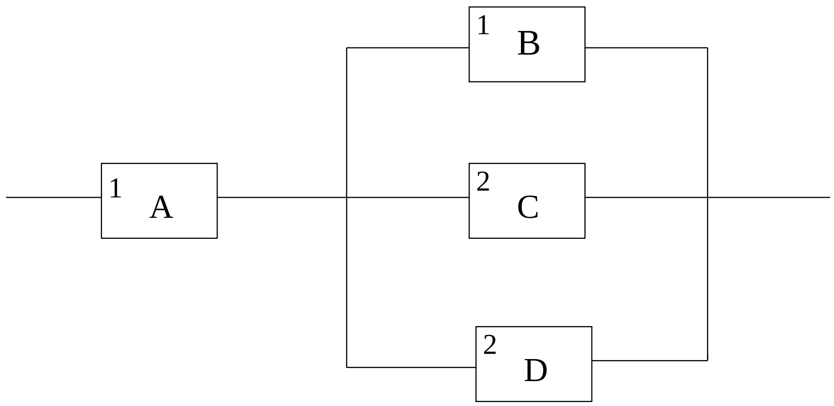

Figure 1: System with four components of two types

l_1	l_2	Φ	Φ_C
0	0	0	0
0	1	0	0
0	2	0	0
1	0	0	0
1	1	1/2	1
1	2	1/2	1
2	0	1	1
2	1	1	1
2	2	1	1

Table 1: $\Phi(l_1, l_2)$ and $\Phi_C(l_1, l_2)$ for the system in Figure 1

structure function with this swap applied if needed, changes from value 0 to 1 for three values of the state vector $\underline{x}$ (with entries alphabetically ordered): $(0,1,0,1), (0,1,1,0), (0,1,1,1)$, as in these cases the failed component A will be replaced by component B which is functioning, and indeed at least one more component functions. The corresponding survival signatures, $\Phi(l_1, l_2)$ for the system without the swap, and $\Phi_C(l_1, l_2)$ with this specific swap applied if needed, are given in Table 1 for all l_1, $l_2 \in \{0, 1, 2\}$.

Let the CDFs of the failure times of the type 1 and 2 components be $F_1(t)$ and $F_2(t)$, respectively. Then the survival function for the system failure time T_S without the swap being possible, and the system failure time T_S^* with the swap applied if needed, are

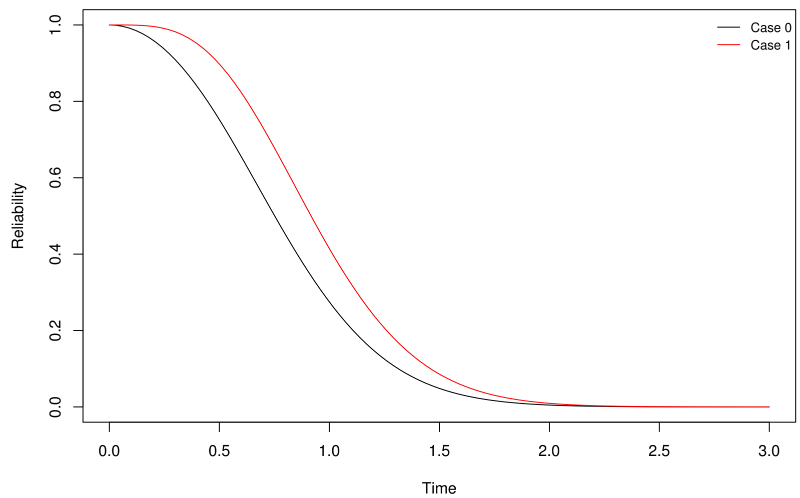

Figure 2: Reliability of system in Figure 1 in case 0 and 1

$$P(T_S > t) = [F_1(t)][1 - F_1(t)][1 - [F_2(t)]^2] + [1 - F_1(t)]^2$$

and

$$P(T_S^* > t) = 2[F_1(t)][1 - F_1(t)][1 - [F_2(t)]^2] + [1 - F_1(t)]^2$$

Figure 2 presents these system survival functions if the failure times of type 1 components have a Weibull distribution with shape parameter 2 and scale parameter 1, $F_1(t) = 1-e^{-t^2}$, and the failure times of type 2 components have an exponential distribution with expected value 1, $F_2(t) = 1 - e^{-t}$. We refer to the case without the swap being possible as 'case 0' and the case with the swap applied if needed as 'case 1'. This figure clearly presents the gain in reliability of the system due to the possible component swap.

2.2. Example 2

The system in Figure 3 consists of 8 components of 3 types, $m = 8$ and $K = 3$. The letters A to H represent the specific components, the numbers 1 to 3 represent the component types. This system consist of three subsystem in series configuration. The first subsystem is a parallel system consisting of components A and D, the second subsystem is a parallel system consisting

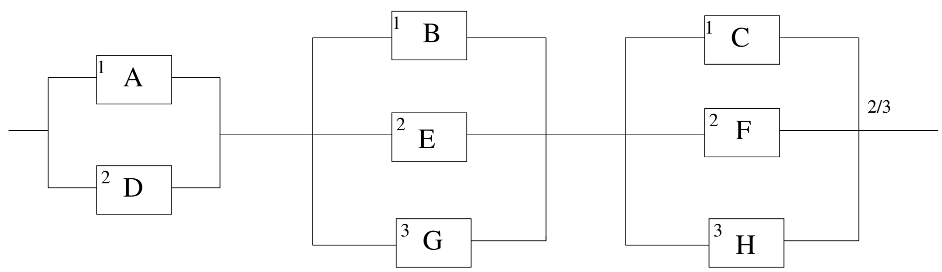

Figure 3: System with 8 components of 3 types; C,F,H form a 2-out-of-3 subsystem

of components B, E and G, and the third subsystem is a 2-out-of-3 system consisting of components C, F and H.

The reliability of this system might be enhancing by a variety of swapping opportunities, we compare two swapping cases. In case 1, we assume that we are able to swap only type 2 components, but we can swap these in any way when needed to keep the system functioning. In case 2, we assume that we are able to swap only type 3 components, which can also be done in both possible ways. We refer to the case without any swapping being possible as case 0.

With $m_1 = 3$, $m_2 = 3$ and $m_3 = 2$, $\Phi(l_1, l_2, l_3)$ in case 0 and $\Phi_C(l_1, l_2, l_3)$ for case 1 and for case 2 are given in Table 2 for all l_1, $l_2 \in \{0, 1, 2, 3\}$ and $l_3 \in \{0, 1, 2\}$.

In order to see the change to the system's reliability as a result of each of swapping case, we assume that the failure times of type 1 components have a Weibull distribution with shape parameter 2 and scale parameter 1, the failure times of type 2 components have an exponential distribution with expected value 1 and the failure times of type 3 components have an exponential distribution with expected value 2, so $F_1(t) = 1 - e^{-t^2}$, $F_2(t) = 1 - e^{-t}$ and $F_3(t) = 1 - e^{-t/2}$. The resulting survival functions of the system failure times are given in Figure 4. Clearly, while both possible swapping regimes of case 1 and case 2 would enhance the system reliability, case 1 provides better improvement than case 2, which is mainly due to the fact that in case 1 three components can be involved in the swaps, including one component in the first subsystem.

2.3. Alternative approach

In our approach in this paper, the effect of a defined swapping regime is fully taken into account through the system structure function, and hence the

l_1	l_2	l_3	Case 0	Case 1	Case 2	l_1	l_2	l_3	Case 0	Case 1	Case 2
0	0	0	0	0	0	2	0	0	0	0	0
0	0	1	0	0	0	2	0	1	0	0	0
0	0	2	0	0	0	2	0	2	1/3	1/3	1/3
0	1	0	0	0	0	2	1	0	0	0	0
0	1	1	0	0	0	2	1	1	2/9	2/3	4/9
0	1	2	0	0	0	2	1	2	5/9	1	5/9
0	2	0	0	0	0	2	2	0	1/3	2/3	1/3
0	2	1	0	0	0	2	2	1	1/2	5/6	7/9
0	2	2	1/3	1	1/3	2	2	2	7/9	1	7/9
0	3	0	0	0	0	2	3	0	2/3	2/3	2/3
0	3	1	1/2	1/2	1	2	3	1	5/6	5/6	1
0	3	2	1	1	1	2	3	2	1	1	1
1	0	0	0	0	0	3	0	0	0	0	0
1	0	1	0	0	0	3	0	1	1/2	1/2	1
1	0	2	0	0	0	3	0	2	1	1	1
1	1	0	0	0	0	3	1	0	1/3	1	1/3
1	1	1	0	0	0	3	1	1	2/3	1	1
1	1	2	2/9	2/3	2/9	3	1	2	1	1	1
1	2	0	0	0	0	3	2	0	2/3	1	2/3
1	2	1	2/9	2/3	4/9	3	2	1	5/6	1	1
1	2	2	5/9	1	5/9	3	2	2	1	1	1
1	3	0	1/3	1/3	1/3	3	3	0	1	1	1
1	3	1	2/3	2/3	1	3	3	1	1	1	1
1	3	2	1	1	1	3	3	2	1	1	1

Table 2: $\Phi(l_1, l_2, l_3)$ and $\Phi_C(l_1, l_2, l_3)$ of system in Figure 3

survival signature. This has the important advantage that each components remains of the same type when compared to the system without swaps being possible. One can interpret this as the component changing location in the system when being swapped.

There is an interesting aspect to this, which we did not encounter in the literature, namely the difference between considering a component in the way described above and the possibility to define a component based on the location. For example, for the simple system in Figure 1, the latter approach would call the location A the 'component A', and if a swap could take place as considered in Example 1, then this 'location-component A' would have as failure time the maximum of the failure times of the two type 1 components

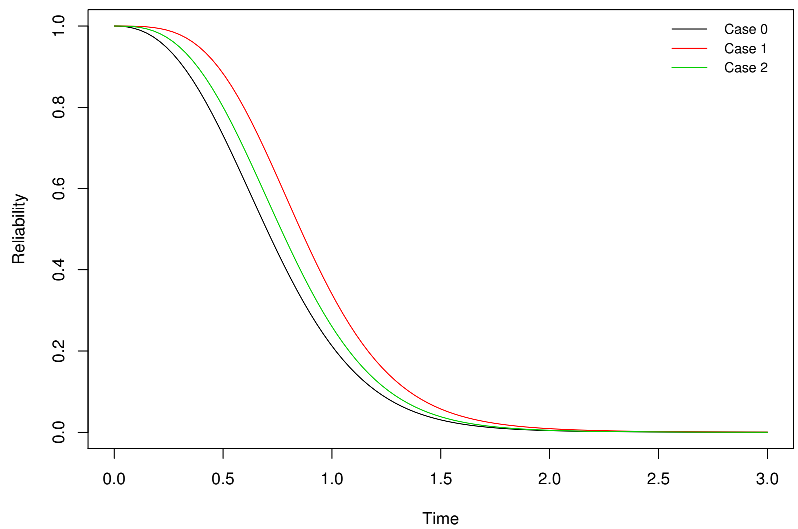

Figure 4: Reliability of the system in Figure 3 in case 0,1 and 2

and the 'location-component B' would have as failure time the minimum of these two failure times. Hence the 'location-components A and B' would not have exchangeable failure times anymore, and hence they would not be of the same type. Already for quite straightforward swapping regimes, this approach would lead to very complicated failure time distributions for the 'location-components', and particularly where possible swaps could depend on other swaps already having been performed, it would quickly become intractable. So, it is crucial to define a component as actually the item that can change location and role in the system, as done in this paper, which lead to a quite straightforward approach.

3. Component reliability importance

We examine the reliability importance of a specific component if we assume that some components in the system can be swapped. We consider the relative importance index $RI_i(t)$ as introduced by [21], which is the difference between the probability that the system functions at time t given that component i functions at time t, and the probability that the system functions at time t given that component i is not functioning at time t, so

		A		B				C,D	
l_1	l_2	$\tilde{\Phi}_1$	$\tilde{\Phi}_0$	$\tilde{\Phi}_1$	$\tilde{\Phi}_0$	l_1	l_2	$\tilde{\Phi}_1$	$\tilde{\Phi}_0$
0	0	0	0	0	0	0	0	0	0
0	1	1	0	0	0	0	1	0	0
0	2	1	0	0	0	1	0	1/2	0
1	0	1	0	1	0	1	1	1/2	1/2
1	1	1	0	1	1	2	0	1	1
1	2	1	0	1	1	2	1	1	1

Table 3: $\tilde{\Phi}_1\,(l_1, l_2)$ and $\tilde{\Phi}_0\,(l_1, l_2)$ for components A, B, C, D

$$RI_i(t) = P(T_S > t|T_i > t) - P(T_S > t|T_i < t)$$

The conditional survival functions $P(T_S > t|T_i > t)$ and $P(T_S > t|T_i < t)$ can be obtained quite easily by deriving the survival signatures corresponding to the two possible states of component i. We can compute this for the system without swapping being possible as well as for specific swapping regimes, and it is of interest to consider the change in importance of specific components resulting from the swapping possibilities. We illustrate this using the same systems and swapping regimes considered in Examples 1 and 2.

3.1. Example 3

Consider again the system in Figure 1 and the same swapping possibility as discussed in Example 1. To calculate the relative importance indices in case 0, in which there is no swapping possible between components, we first calculate the survival signature of the system conditional on the component of interest functioning, which we denote by $\tilde{\Phi}_1\,(l_1, l_2)$, where it should be noted that either l_1 or l_2 (corresponding to the type of the component of interest) now only takes values in $\{0, \ldots, m_k - 1$ for $k = 1$ or $k = 2$, as it only reflects the number of the other components of the same type that are functioning. Similarly, we calculate the survival signature of the system conditional on the component of interest not functioning, which we denote by $\tilde{\Phi}_0\,(l_1, l_2)$. The survival signatures $\tilde{\Phi}_1\,(l_1, l_2)$ and $\tilde{\Phi}_0\,(l_1, l_2)$ are given in Table 3 for all components, note of course that these are identical for components C and D.

		A		B				C,D	
l_1	l_2	$\tilde{\Phi}_{C_1}$	$\tilde{\Phi}_{C_0}$	$\tilde{\Phi}_{C_1}$	$\tilde{\Phi}_{C_0}$	l_1	l_2	$\tilde{\Phi}_{C_1}$	$\tilde{\Phi}_{C_0}$
0	0	0	0	0	0	0	0	0	0
0	1	1	0	1	0	0	1	0	0
0	2	1	0	1	0	1	0	1	0
1	0	1	0	1	0	1	1	1	1
1	1	1	1	1	1	2	0	1	1
1	2	1	1	1	1	2	1	1	1

Table 4: $\tilde{\Phi}_{C_1}(l_1, l_2)$ and $\tilde{\Phi}_{C_0}(l_1, l_2)$ for components A, B, C, D

The relative importance index for component A, $RI_A(t)$, is derived by

$$RI_A(t) = \sum_{l_1=0}^{1} \sum_{l_2=0}^{2} \left[\tilde{\Phi}_1(l_1, l_2) - \tilde{\Phi}_0(l_1, l_2) \right] \prod_{k=1}^{2} P(C_t^k = l_k)$$

leading to

$$RI_A(t) = [F_1(t)] \left[1 - [F_2(t)]^2 \right] + [1 - F_1(t)]$$

Similarly, we derive

$$RI_B(t) = [1 - F_1(t)][F_2(t)]^2$$

$$RI_C(t) = RI_D(t) = [F_1(t)][1 - F_1(t)][F_2(t)]$$

We aim to determine the differences that might occur in $RI_A(t)$, $RI_B(t)$, $RI_C(t)$ and $RI_D(t)$ in case 1, in which we assume that if the component A fails while the component B still functions, component A will be swapped by component B. To calculate the relative importance indices in case 1, $\tilde{\Phi}_{C_1}(l_1, l_2)$ represents the survival signature with the swap enabled, if the component of interest functions, and $\tilde{\Phi}_{C_0}(l_1, l_2)$ if the component does not function. We denote the relative importance index of component i if the swap between components A and B is possible by $RI_i^*(t)$. Table 4 presents $\tilde{\Phi}_{C_1}(l_1, l_2)$ and $\tilde{\Phi}_{C_0}(l_1, l_2)$ for all the components in case 1.

The relative importance index for component A, $RI_A^*(t)$, is derived by

$$RI_A^*(t) = \sum_{l_1=0}^{1} \sum_{l_2=0}^{2} \left[\tilde{\Phi}_{C_1}(l_1, l_2) - \tilde{\Phi}_{C_0}(l_1, l_2) \right] \prod_{k=1}^{2} P(C_t^k = l_k)$$

leading to

$$RI_A^*(t) = [F_1(t)] \left[1 - [F_2(t)]^2 \right] + [1 - F_1(t)][F_2(t)]^2$$

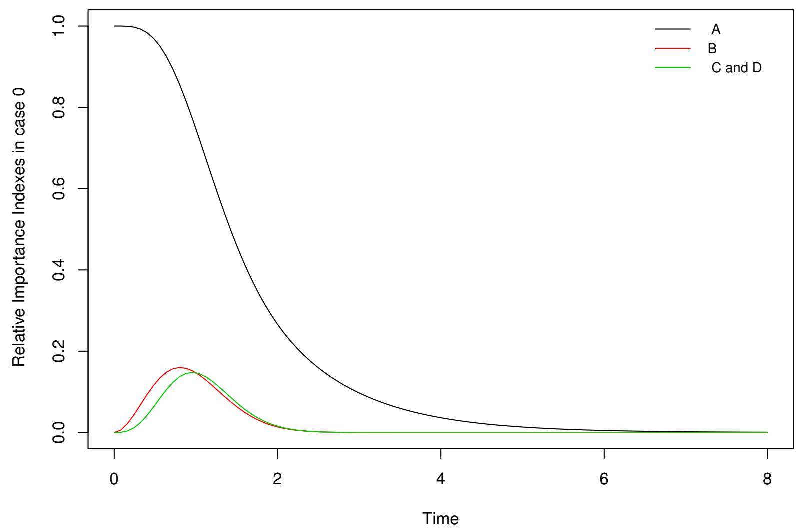

Figure 5: The relative importance indices of components in Figure 1 in case 0.

Similarly we derive

$$RI_B^*(t) = [F_1(t)]\left[1 - [F_2(t)]^2\right] + [1 - F_1(t)][F_2(t)]^2$$

$$RI_C^*(t) = RI_D^*(t) = 2[F_1(t)][1 - F_1(t)][F_2(t)]$$

To compare the relative importance indices of the system's components in case 0 and 1, we use the same failure time distributions for type 1 and type 2 components as in Example 1. Figures 5 and 6 show the relative importance indices of the system's components in cases 0 and 1, respectively. These figures show that in case 0 component A is clearly most important, yet with the swapping possible between components A and B in case 1 these two components become equally important.

3.2. Example 4

We consider the component importance for the system in Figure 3, under the two swapping cases that we introduced in Example 2, namely in case 1 we are able to swap type 2 components and in case 2 we are able to swap type 3 components. We refer to the original case in which there is no swap option

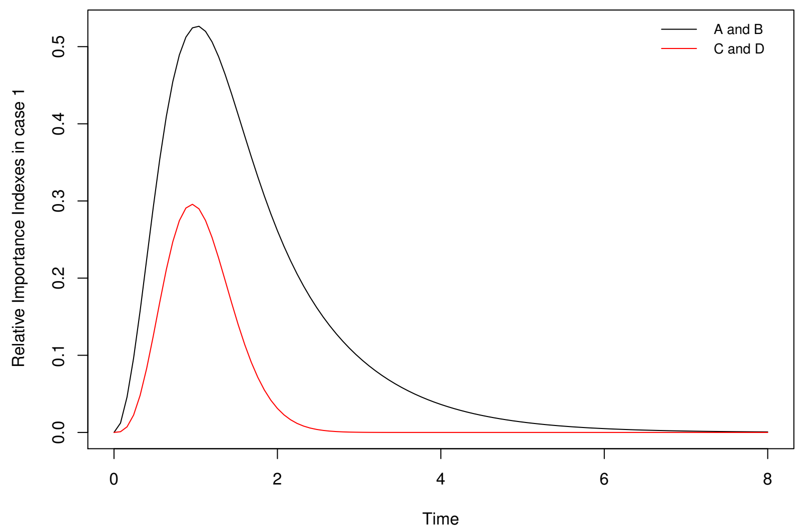

Figure 6: The relative importance indices of components in Figure 1 in case 1.

as case 0. We also assume the same component failure time distributions as in Example 2.

Figures 7, 8 and 9 show the relative importance indices of the components in case 0, case 1 and case 2 respectively. In case 1, the components of type 2, so D, E and F, become equally important and, in case 2, the components of type 3, so H and G, become equally important for system reliability as a result of the ability of swapping among them. These figures also show that the importance of specific components is dependent on the swapping cases that would be allowed as well as the component failure time distributions. If we consider only the period of time from $t = 0$ to $t = 0.4$, for example, then we can see that in this period, in case 0, component H is the most important component, in case 1, component C becomes the most important component, and in case 2, component A becomes the most important component for system reliability.

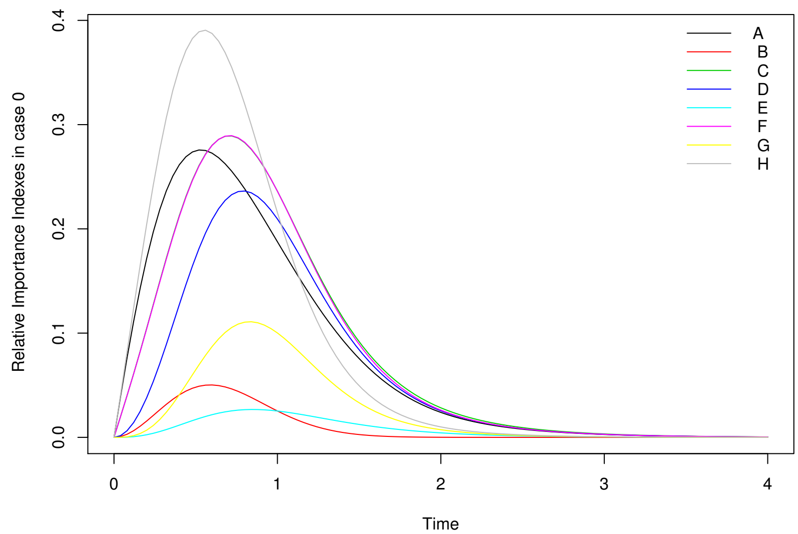

Figure 7: The relative importance indices of components in Figure 3 in case 0.

4. Joint reliability importance (JRI)

We consider the joint reliability importance index JRI of components i, j, given by following equation

$$\begin{aligned} JRI_{i,j}(t) &= P(T_S > t | T_i > t, T_j > t) - P(T_S > t | T_i > t, T_j \leq t) \\ &\quad - P(T_S > t | T_i \leq t, T_j > t) + P(T_S > t | T_i \leq t, T_j \leq t) \end{aligned}$$

for $t > 0$ [19]. The joint reliability importance JRI is a measure of interaction of the two components in a system with regard to their contribution to the system reliability. The value of JRI indicates that one component is more or less important, or has the same importance, when the other is functioning. If $JRI > 0$ then one component becomes more important when the other is functioning (so they can be regarded as 'complements'). If $JRI < 0$ then one component becomes less important when the other is functioning ('substitutes'), while if $JRI = 0$ then one components importance is unchanged by the functioning of the other [19]. We consider again the influence of possible swaps on the joint reliability importance of components. The importance measure and approach considered in this section can be generalized

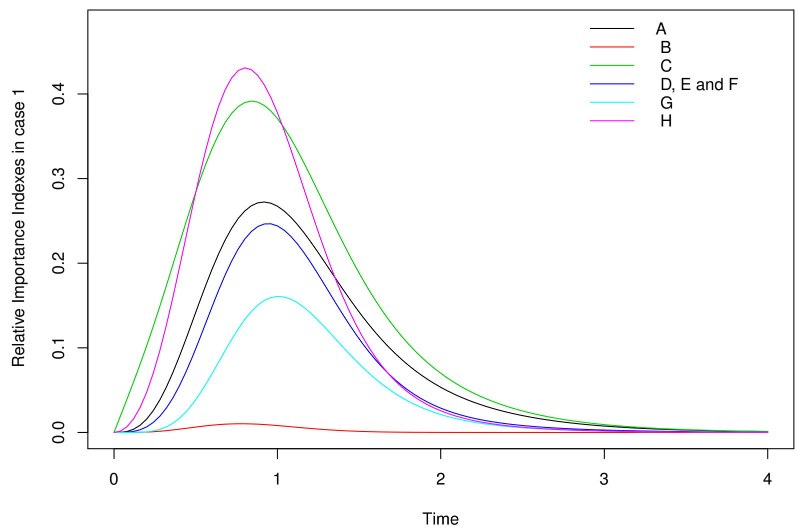

Figure 8: The relative importance indices of components in Figure 3 in case 1.

quite straightforwardly to joint importance of more than two components, but this tends to be of less practical relevance. Computing the conditional survival functions given the state of two components is again quite straightforward, and requires the computation of the corresponding survival signatures. We illustrate this using the same two systems and scenarios considered in Examples 1 and 2, and also in Examples 3 and 4.

4.1. Example 5

We consider the JRI of each pair of components in Figure 1 for the same swapping case introduced in Example 1. The joint reliability importance of components A and B in case 0, in which there is no swapping possible, is denoted by $JRI_{A,B}$. Note that, given the states of these two components, the only variable left is the number of functioning components of type 2, so components C and D, hence we can represent the survival signatures given the states of components A and B as a function of only l_2, the number of functioning components of type 2. Table 5 presents the survival signatures $\tilde{\Phi}_{1,1}(l_2)$, $\tilde{\Phi}_{1,0}(l_2)$, $\tilde{\Phi}_{0,1}(l_2)$ and $\tilde{\Phi}_{0,0}(l_2)$, where the first subscript represents the state of component A and the second the state of component B.

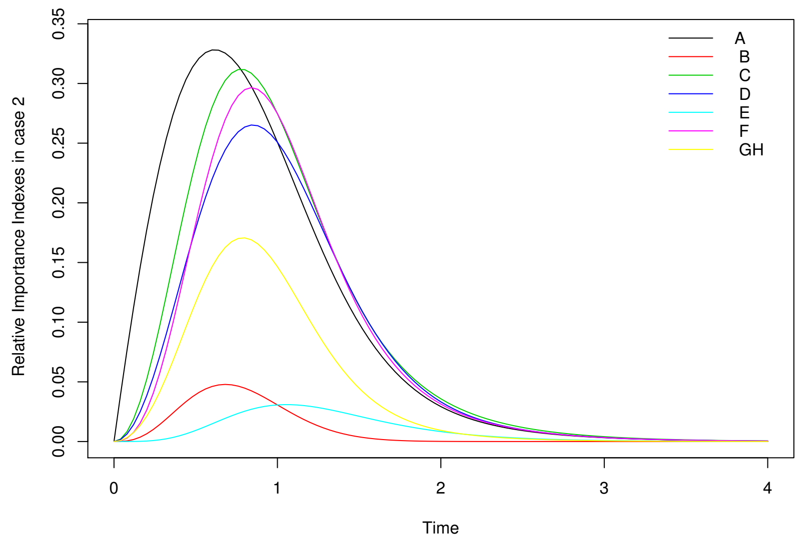

Figure 9: The relative importance indices of components in Figure 3 in case 2.

l_2	$\tilde{\Phi}_{1,1}$	$\tilde{\Phi}_{1,0}$	$\tilde{\Phi}_{0,1}$	$\tilde{\Phi}_{0,0}$
0	1	0	0	0
1	1	1	0	0
2	1	1	0	0

Table 5: Survival signatures given states of components A and B

The JRI for components A and B can be derived by

$$JRI_{A,B}(t) = \sum_{l_2=0}^{2} \left[\Phi_{1,1}(l_2) - \tilde{\Phi}_{1,0}(l_2) - \tilde{\Phi}_{0,1}(l_2) + \tilde{\Phi}_{0,0}(l_2) \right] P(C_t^2 = l_2)$$

leading to

$$JRI_{A,B}(t) = [F_2(t)]^2$$

By the same method we derive

$$JRI_{A,C}(t) = JRI_{A,D}(t) = [F_1(t)][F_2(t)]$$

$$JRI_{B,C}(t) = JRI_{B,D}(t) = -[1 - F_1(t)][F_2(t)]$$

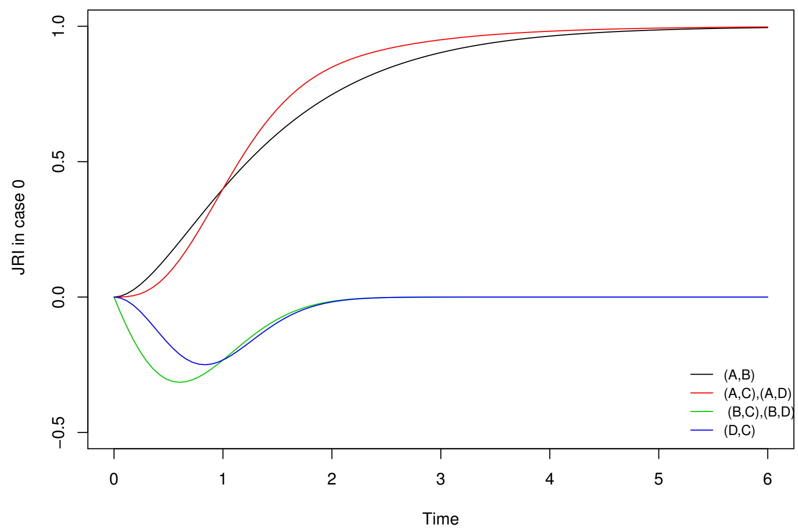

Figure 10: JRI of each pair of components in Figure 1, case 0.

$$JRI_{D,C}(t) = -[F_1(t)][1 - F_1(t)]$$

These joint reliability indices are presented in Figure 10, where the same component failure time distributions have been assumed as in Example 1. These indices will be compared to the similar indices in case of component swaps being possible later in this example.

We now consider the same possible component swap as in Example 1, that is component B can take over the role of component A if needed. Let $JRI^*_{i,j}(t)$ denote the joint reliability importance of components i and j in case 1, so with this swap being possible. To calculate $JRI^*_{i,j}(t)$, we first compute the four survival signatures corresponding to this swap in case 1 and conditioned on the respective states for components i, j. This leads to the following results

$$JRI^*_{A,B}(t) = \left[2[F_2(t)]^2\right] - 1$$

$$JRI^*_{A,C}(t) = JRI^*_{A,D}(t) = JRI^*_{B,C}(t) = JRI^*_{B,D}(t) = [2F_1(t) - 1][F_2(t)]$$

$$JRI^*_{D,C}(t) = -2[F_1(t)][1 - F_1(t)]$$

Again assuming the same component failure time distributions as in Example 1, the resulting system failure time survival functions are given in Figure 11

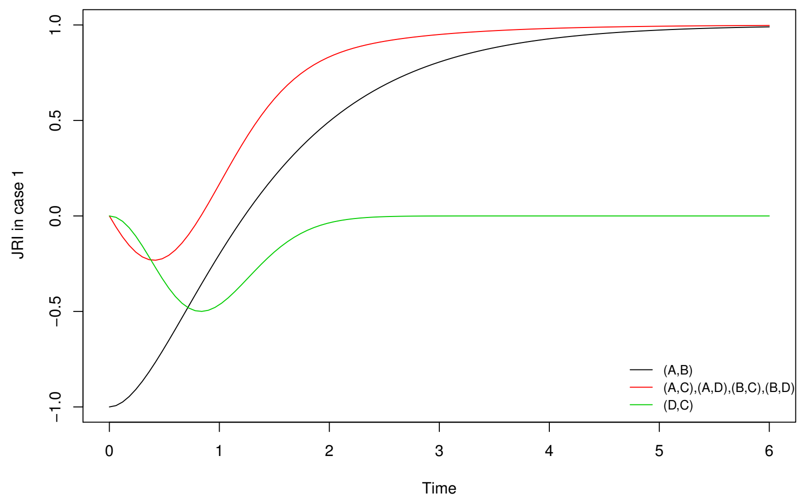

Figure 11: JRI of each pair of components in Figure 1, case 1.

Figures 10 and 11 illustrate the JRIs in case 0 and 1, respectively. In case 0, the pairs of components (A, B), (A, C) and (A, D) are each complementary, while (B, C), (B, D) and (D, C) are substitutes. It is clear by comparing these figures that the interaction of each pair of components with regard to their contribution to the system reliability is impacted by component swapping being possible. In particular, not all pairs are complements or substitutes for all t anymore, where particularly the joint reliability importance of the pair (A, B) is much affected by the swapping opportunity.

4.2. Example 6

For the system in Figure 3, discussed in Examples 2 and 4, there are 28 pairs of component. We only briefly illustrate joint reliability importance for this system, by considering the JRI for components G and H in the three cases considered before, namely case 0 of no swaps being possible, case 1 where components D, E, F (type 2) can be swapped, and case 2 where components G and H (type 3) can be swapped. With the same component failure time distributions assumed as in Example 2, Figure 12 presents these three JRIs. In case 0 the components G and H are complementary. The possible swapping in case 1 has the effect that components G and H become reliability substitutes. In case 2, in which we able to swap these two components with

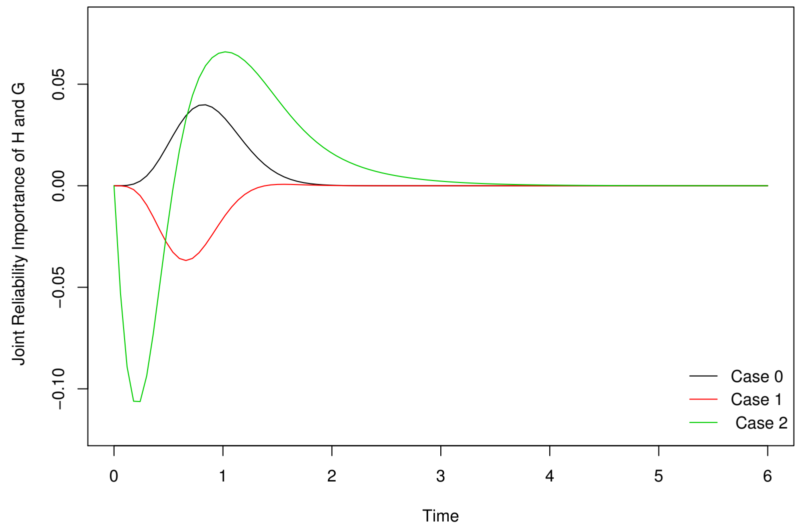

Figure 12: JRI for components G and H, three cases

each other, they become reliability complements until a specific time point and they become reliability substitutes after that time, of course the precise times involved depend on the failure time distributions of all components.

5. Concluding remarks

In this paper we have considered quantification of system reliability if some components can be swapped. It is crucial that this is a different activity than popular, and well-studied approaches such as the use of additional components to provide increased redundancy, the use of standby components, maintenance activities or increased component reliability. It is likely to be attractive to consider a component swap, upon failure of a critical component, if this activity can be done at low cost and if it may e.g. be seen as a temporary measure in order to prepare repairs.

We implemented the survival signature concept that was introduced by [9] in this study. We considered component importance, which was particularly simplified by the use of the survival signature. Further research related to this approach will address several questions. We aim to include considerations of costs, both to prepare for possible swaps and to execute swaps, and study the contribution swaps can make to system resilience in comparison to other activities, including more in-built redundancy, standby components, or maintenance and replacement activities. We also wish to consider other importance measures. A further interesting topic for future research is the possibility to swap components when they are all still functioning. This could be attractive if one has the opportunity to swap components of different types, where for example a critical component may, while still functioning, be swapped with another component at a certain time if they have different hazard rates over time, for example a component with increasing hazard rate may be best to use in a critical part of the system in early stages, to then be swapped by a component with decreasing hazard rate to improve system reliability at later stages.

The effect of the swapping of components is entirely reflected through the change in the survival signature. It may be of interest to investigate whether or not this change can also be reflected by a distortion of the component reliabilities [14], which may provide a further tool for comparison of different systems and different swapping routines. It has been shown that very efficient simulation methods can be based on the survival signature [15], the same simulation method can perhaps also be used to only learn about difference in reliability for two swapping regimes. In ongoing research, the authors are including cost considerations for enabling or actually executing swaps, and costs corresponding to system down-time, to decide on optimal swapping regimes.

The approach presented in this paper requires repeated calculation of survival signatures. [22] has created a function in the statistical software R to compute the survival signature, given a graphical presentation of the system structure. This will be necessary for our work for systems that are not very small, and it will be of interest to create a tool that can automatically compute all the survival signatures required in case of a substantial system with many component swapping opportunities.

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